vector g^0 and use the Gram-Schmidt method to construct the system of $n-1$ orthonormed vectors. Further, we add to these n orthogonal vectors the same number of vectors of opposite direction and then use all $2n$ vectors as initial vectors (the plane case is illustrated in Fig.5). In numerical experiments carried out for $n=2$ we found no cases in which a set of local maxima obtained in this manner did not contain a global maximum. The algorithm converges to a local maximum after 2-4 iterations irrespective of the dimensions of the system (up to $n=6$ in the experiments), and this is at least twice as fast as in the case of similar methods /2/ of searching for the minimum of a functional.

In conclusion, we note that the proposed algorithm can be applied to non-linear systems including the case with a non-convex domain of attainability.

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ON THE CONSTRUCTION OF GENERAL SOLUTIONS OF THE THEORY OF THE ELASTICITY OF INHOMOGENEOUS SOLIDS*

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The elasticity theory equations are decomposed in the case when the shear modulus is a function of one Cartesian coordinate while Poisson's ratio is a function of three coordinates. Such a separation is possible for transverse isotropy when both shear coefficients depend just on the coordinates of the normal isotropy plane. It is assumed that the mass forces are potential.

Decomposition of the elasticity theory equations of an isotropic body by extraction of the normal rotation deformation /1/ was later extended to the case of a transversely-isotropic body /2/. Such a separation was performed for an isotropic body /3/ and for a transversely-isotropic body /4/* (*See also Puro, A.E., Some Exact Particular Solutions of the Statics Equations of an Inhomogeneous Medium, Candidate Dissertation, Tallinn, 1975.) for a one-dimensional inhomogeneity when the elasticity coefficients depend on one Cartesian coordinate.

1. A transversely-isotropic body is referred to a rectangular Cartesian system of coordinates and the z-axis is perpendicular to the plane of body isotropy.

We consider both shear coefficients $c_{44}=c^{-1}$, $(c_{11}-c_{12})/2=G$ in the generalized Hooke's law

$$\sigma_{xx}=c_{11}\varepsilon_{xx}+c_{12}\varepsilon_{yy}+c_{13}\varepsilon_{zz},\quad \sigma_{xy}=(c_{11}-c_{12})\varepsilon_{xy}$$
$$\sigma_{yy}=c_{12}\varepsilon_{xx}+c_{11}\varepsilon_{yy}+c_{13}\varepsilon_{zz},\quad \sigma_{xz}=2c_{44}\varepsilon_{xz}$$
$$\sigma_{zz}=\sigma=c_{13}(\varepsilon_{xx}+\varepsilon_{xy})+c_{33}\varepsilon_{zz},\quad \sigma_{yz}=2c_{44}\varepsilon_{yz}$$

differential functions of just the z coordinate while the remaining elasticity coefficients c_{ik} are functions of three coordinates. It is also assumed that the mass force vector $\mathbf{M}$ and the displacement vector $\mathbf{u}$ are decomposed into potential and solenoidal components in the plane of isotropy and expressed, respectively, in terms of the potentials

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$$\mathbf{M}=\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\Omega}{\partial y}\right)\mathbf{i}+\left(\frac{\partial\varphi}{\partial y}-\frac{\partial\Omega}{\partial x}\right)\mathbf{j}+\frac{\partial\chi}{\partial z}\mathbf{k}$$
$$\mathbf{u}=\left(\frac{\partial F}{\partial x}+\frac{\partial N}{\partial y}\right)\mathbf{i}+\left(\frac{\partial F}{\partial y}-\frac{\partial N}{\partial x}\right)\mathbf{j}+w\mathbf{k}$$

Substituting **M** and **u** into the equilibrium equation $\operatorname{div}\sigma+\mathbf{M}=0$, we obtain a system of three equations that are separated into an equation for the normal rotation potential N (a solution of the second kind)

$$\left[G\Delta_{+}+\frac{\partial}{\partial z}c_{44}\frac{\partial}{\partial z}\right]N=-\Omega \tag{1.1}$$

and a system of two connected equations (a solution of the first kind) in w and F

$$\frac{\partial}{\partial z}\left[c_{44}\left(w+\frac{\partial F}{\partial z}\right)\right]+c_{13}\frac{\partial w}{\partial z}+c_{11}\Delta_{+}F=-\varphi \tag{1.2}$$
$$\Delta_{+}\left[c_{44}\left(w+\frac{\partial F}{\partial z}\right)\right]+\frac{\partial}{\partial z}\left[c_{33}\frac{\partial w}{\partial z}+c_{13}\Delta_{+}F\right]=-\frac{\partial\chi}{\partial z} \tag{1.3}$$
$$(\Delta_{+}F=(\partial^2/\partial x^2+\partial^2/\partial y^2)F)$$

The method of separating the equations is identical to that mentioned earlier /2, 3/ (see also the dissertation mentioned in the footnote) and, consequently, will not be presented here.

An analogous separation of the equations is possible in the dynamic case if the density of the medium is a function of just the z coordinate.

We will present the derivation of other kinds of resolving equations of solutions of the first kind. To do this we introduce the function $L(x, y, z)$ by setting

$$w=-(c\partial L/\partial z+\partial F/\partial z) \tag{1.4}$$
$$\sigma_{zz}=c_{33}\partial w/\partial z+c_{13}\Delta_{+}F=-\Delta_{+}L-\chi \tag{1.5}$$

Eq.(1.3) is satisfied identically. Substituting the relationship (1.4) into (1.2) and (1.5), we eliminate w from these equations. We will write the system obtained for the resolving functions in the form

$$\partial^2F/\partial z^2+(c-b)\,\partial^2L/\partial z^2+c'\partial L/\partial z+a\Delta_{+}L=a\chi-b\varphi \tag{1.6}$$
$$\Delta_{+}F-d\partial^2L/\partial z^2+b\Delta_{+}L=b\chi-d\varphi \tag{1.7}$$
$$(\beta_{11}=d=c_{33}\rho,\ \beta_{13}=-b=c_{13}/\rho,\ \beta_{33}=c_{11}/\rho,\ \rho=c_{11}c_{33}-c_{13}^2)$$

For brevity here, the elastic constants presented /4/ β_{ik} are denoted in the manner mentioned above.

System (1.6) and (1.7) is mixed in the sense that the stress function L and the displacement potential F are not known.

We will examine the case of the homogeneous equations $\varphi=\chi=0$ (no mass forces) in greater detail. The stress components of solutions of the first kind are expressed, when (1.6) and (1.7) are taken into account, by the formulas

$$\sigma_{xx}=\partial^2L/\partial z^2-\partial^2\Phi/\partial y^2,\quad \sigma_{yy}=\partial^2L/\partial z^2-\partial^2\Phi/\partial x^2,\quad \sigma_{zz}=\Delta_{+}L \tag{1.8}$$
$$\sigma_{xy}=\partial^2\Phi/\partial x\partial y,\ \sigma_{xz}=-\partial^2L/\partial x\partial z,\ \sigma_{yz}=-\partial^2L/\partial y\partial z\quad (\Phi=[c_{11}-c_{12}]F)$$

It follows from relations (1.8) /5/ that L and Φ are diagonal elements of the stress function tensor ψ whereas its remaining elements equal zero: $\psi=\operatorname{diag}\{L, L, -\Phi\}$. Therefore, (1.6) and (1.7) can be obtained from the compatibility equations for the stress tensor expressed by means of (1.8). We obtain the equation in L by eliminating F from (1.6) and (1.7)

$$\Delta_{+}\left(a\Delta_{+}+\frac{c-2b}{2}\frac{\partial^2}{\partial z^2}\right)L+\frac{\partial^2}{\partial z^2}\left(d\frac{\partial^2}{\partial z^2}-\frac{c-2b}{2}\Delta_{+}\right)L-c''\Delta_{+}L=0 \tag{1.9}$$

Let us examine the connection of the functions L, F with the general solutions obtained earlier.

If b and d depend only on z we find from (1.7) by the substitution $L=\Delta_{+}L_0$

$$F=d\partial^2L_0/\partial z^2-b\Delta_{+}L_0$$

Substituting F into (1.6), we arrive at (1.9) for L_0 /4, 5/ and we find w from (1.4)

$$w=\partial[b\Delta_{+}L_0-d\partial^2L_0/\partial z^2]/\partial z-c\partial\Delta_{+}L_0/\partial z$$

If d and b are independent of z while $c(z) = c_1 + c_2 z$ is a linear function in z, then by the substitution $L = \partial^2 L_+/\partial z^2$ we find F from (1.6)

$$F = (b - c)\,\partial^2 L_+/\partial z^2 + c'\partial L_+/\partial z - a\Delta_+ L_+$$

Substituting F into (1.7), we again arrive at (1.9) for L_+, where

$$w = [a\Delta_+ - b\partial^2/\partial z^2]\partial L/\partial z$$

The displacement components in the last two cases are expressed by means of third-order derivatives of the resolving functions if the c_{ik} are independent of z, then the order of the derivatives can be reduced to the second.

In the case, system (1.6) and (1.7) can be solved for $\partial^2 L/\partial z^2$, $\Delta_+ L$

$$\frac{\partial^2 L}{\partial z^2} = \left[b^0 \frac{\partial^2 F}{\partial z^2} - a^0 \Delta_+ F\right], \quad \Delta_+ L = \left[d^0 \frac{\partial^2 F}{\partial z^2} - (b^0 - c^0)\Delta_+ F\right] \tag{1.10}$$

$$(a^0 = c_{11}/\tau, \quad b^0 = -c_{13}/\tau, \quad d^0 = -c_{33}/\tau, \quad c^0 = -\rho\,\tau, \quad \tau = 1 + c_{33}/c)$$

Defining $F = \partial L_*/\partial z$ and substituting F into the first relationship of (1.10), we find

$$\frac{\partial L}{\partial z} = \left[b^0 \frac{\partial^2 L_*}{\partial z^2} - a^0 \Delta_+ L_*\right]$$

We obtain the resolving equation for L_* by differentiating the second relationship of (1.10) with respect to z and substituting F and $\partial L/\partial z$ into this expression. Outwardly it is identical with (1.9) except that all the coefficients a, b, c, d are replaced by appropriate coefficients with zero subscript and there is no last component (in this case the coefficient c is considered constant). The displacement vector components equal, respectively

$$u_{x,y} = \frac{\partial}{\partial x, \partial y}\left(\frac{\partial}{\partial z} L_*\right), \quad w = \frac{-1}{c_{44} + c_{33}}\left[c_{44}\frac{\partial^2 L_*}{\partial z^2} + c_{11}\Delta_+ L_*\right]$$

Lekhnitskii first obtained such a representation for a homogeneous medium in the axially symmetric case and it was generalized in /2/.

We will write the system of equations in symmetric form by using the substitution $F = F_0 - cL/2$

$$2\Delta_1 F_0 = -\sqrt{D}\Delta_1 L + c''L, \quad 2\Delta_2 F_0 = \sqrt{D}\Delta_2 L + c''L \tag{1.11}$$

Here

$$\Delta_i F_0 = (m_i^2 \Delta_+ + \partial^2/\partial z^2) F_0, \quad i = 1, 2$$

$$m_{1,2}^2 = [(c - 2b) \pm \sqrt{D}]/(2d), \quad dm^4 - (c - 2b)m^2 + a = 0, \quad D = (c - 2b)^2 - 4ad$$

where $m_{1,2}$ are roots of the characteristic equation and D is the discriminant of this equation.

For $c'' = 0$ and $D = \text{const}$ the general solution can be expressed in terms of the sum of two functions L_1, L_2 that, respectively, satisfy the equations $\Delta_i L_i = 0$. Indeed, the expression

$$L(x, y, z) = L_1/\sqrt{D} + L_2/\sqrt{D}; \; F_0(x, y, z) = L_1/2 - L_2/2$$

satisfies (1.11).

The representation obtained for the general solution is identical, apart from constants, with the known solution /6/ used in the case of a homogeneous medium. Hence, depending on the form of the inhomgeneity, the resolving equations of the first kind at more conveniently utilized in some form but in any case (1.9) is actually solved.

2. We will present the fundamental relationships for the case when the roots of the characteristic equation $m_1^2 = m_2^2 = m_0^2$ are degenerate. From the condition $D = 0$ we obtain the connection between the coefficients $c_{13} = \sqrt{c_{11}c_{33}} - 2c_{44}$ where $m_0^2 = \sqrt{c_{11}/c_{33}} = \sqrt{a/d}$.

Since Eqs.(1.12) are identical in this case, we will take (1.7) as the second resolving equation

$$2\Delta_0 F_0 = c''L, \quad \Delta_+ F_0 = d\Delta_0 L \tag{2.1}$$

$$(\Delta_0 F = (m_0^2 \Delta_+ + \partial^2/\partial z^2) F_0)$$

If the coefficients d, m_0 depend only on z, then by introducing the function $L = \Delta_+ L_0$, $F_0 = d\Delta_0 L_0$ we satisfy the second equation in (2.1) while the first equation in (2.1) determines the resolving function L_0

$$2\Delta_0(d\Delta_0 L_0) - c''\Delta_+ L_0 = 0 \tag{2.2}$$

If $c'' \neq 0$, then L is found from the first equation in (2.1)

$$L = 2(c'')^{-1}\Delta_0 F_0 \tag{2.3}$$

while the second equation in (2.1) becomes a resolving equation for F_0

$$2d\Delta_0[(c'')^{-1}\Delta_0 F_0] - \Delta_+ F_0 = 0 \tag{2.4}$$

The equations of an isotropic medium are obtained as a special case from the formulas presented above for $c_{11} = c_{33} = \lambda + 2\mu$ the remaining coefficients take the values $m_0^2 = 1,\ d = c(1-\nu),\ c^{-1} = c_{44} = \mu$.

3. We will examine certain inhomogeneity laws for which the solutions of the resolving Eq.(1.9) can be obtained explicitly.

Under the condition $c'' = 0$, while the coefficients $a,\ (c-2b),\ d$ are proportional to $\exp(\alpha x + \beta y + \gamma z)$ (1.9) reduces to an equation with constant coefficients whose solution can be obtained explicitly.

If $c_{11} = c_{11}(z)/n^2(x, y),\ c_{33} = c_{33}(z)\, n^2(x, y)$, while the remaining coefficients depend only on z the variables in (1.9) are separated partially by introducing $L(x, y, z) = L(z, s)\,\psi(x, y, s)$. We obtain a two-dimensional Helmholtz equation for $\psi(x, y, s)$

$$\Delta_+\psi(x, y, s) + s^2 n^2(x, y)\,\psi(x, y, s) = 0 \tag{3.1}$$

and an ordinary differential equation for $L(z, s)$

$$\left\{\frac{d^2}{dz^2}\left[d\,\frac{d^2}{dz^2} - \frac{c-2b}{2}s^2\right] - s^2\left[\frac{c-2b}{2}\,\frac{d^2}{dz^2} - as^2 - \frac{c''}{2}\right]\right\} L = 0 \tag{3.2}$$

For complete separation of the variables it is necessary that the variables in (3.1) be separated. To investigate this question, we will write (3.1) in a curvilinear orthogonal system of coordinates $\alpha,\ \beta$

$$\left[\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} + s^2\left|\frac{d\zeta}{d\gamma}\right|^2 n^2(\alpha, \beta)\right]\psi(\alpha, \beta) = 0 \tag{3.3}$$

Here $\zeta = x + iy$ is an analytic function of the complex variable $\gamma = \alpha + i\beta$.

The condition for separation of variables is the possibility of the representation /7/

$$|d\zeta/d\gamma|^2 n^2(\alpha, \beta) = n_0(\alpha) + n_1(\beta) \tag{3.4}$$

In this case the solution (3.3) is represented in the form $\psi(\alpha, \beta, s) = \psi_1(\alpha, s, k)\,\psi_2(\beta, s, k)$ where each of the components satisfies the equation (k is a constant)

$$\psi_{1,2}'' + (s^2 n_{0,1} \pm k^2)\,\psi_{1,2} = 0 \tag{3.5}$$

Let us examine the case when (3.5) reduces to a known equation. The replacement $\psi_1(\alpha) = y(t)/\sqrt{t'}$ converts (3.5) into normal form /8/ ($t(\alpha)$ is the new variable)

$$d^2y(t)/dt^2 + |t'|^{-2}[s^2 n_0(t) + k^2 - D(t', \alpha)/2]\, y(t) = 0 \tag{3.6}$$

$$D(t', \alpha) = -2(t')^{1/2} d^2 (t')^{-1/2}/dt^2$$

($D(t', \alpha)$ is the Schwartz derivative). Selecting the known equations in normal form, we determine $t(\alpha)$ and $n_0(\alpha)$ from (3.6), for which (3.5) reduce to (3.6). In turn, from the known $n_0(\alpha),\ n_1(\beta)$ and $\zeta(\gamma)$ the $n^2(\alpha, \beta)$ is found from (3.4) for which the solutions of (3.3) are expressed in terms of the solution of the equations chosen earlier.

Thus, (3.6) reduces to the equation $y'' + m^2 y = 0$ for $t = \alpha,\ n_0(\alpha) = n_0,\ s^2 n_0 + k^2 = m^2$ to the Bessel equation $y'' + [\lambda + (1/4 - \mu^2)/t^2]\, y = 0$ for $t = \alpha + c,\ n_0(\alpha) + n_1/\alpha^2$ and for $t = \exp(\alpha c + c_1),\ n_0(\alpha) = n_0 + n_1 \exp(2\alpha c)$.

The case when (3.6) reduces to the Whittaker equation, the hypergeometric equation, can be mentioned.

Application of this method to the solution of analogous problems of electrodynamics is examined in greater detail in /7/.

We will now examine those inhomogeneity laws for which an explicit solution can be obtained from (3.2).

For an inhomogeneity of the form $a(z) = az^n,\ b(z) = bz^{n+2},\ c(z) = cz^{n+2},\ d(z) = dz^{n+4}$ Eq.(3.2) goes over into the Euler equation where $c_{11} \approx z^{-(n+4)},\ c_{33} \approx z^{-n},\ c_{13} \approx z^{-(n+2)}$. The solution is expressed in terms of power functions

$$L(z, s) = \sum_{j=1}^{4} L_j(s)\, z^{\lambda_j}$$

$$\lambda_j = -{}^1/_2(n+1) \pm \{[d(n^2 + 4n + 5) + 2s^2(c - 2b) \pm 2\sqrt{D_1}]/(4d)\}^{1/2}$$

$$D_1 = s^4 D + (n+2)\, d\,[2s^2(c - 2b(n+2)) + (n+2)d]$$

In particular, for $n = -2$ the roots are

$$\lambda_j = {}^1/_2 \pm \sqrt{{}^1/_4 + s^2 m_i^2} \tag{3.7}$$

For an inhomogeneity of the form $a(z) = (az + a_0)e^{\alpha z}$, $b(z) = (bz + b_0)e^{\alpha z}$, $c(z) = (cz + c_0)e^{\alpha z}$, $d(z) = (dz + d_0)e^{\alpha z}$ (3.2) is the Laplace equation /8/. Its solution can be represented in the form of a definite integral

$$L(z, s) = L(s)\int e^{zt}\varphi(t)\,dt$$

where the integration limits are determined by a well-known method /8/. We obtain a first-order equation for $\varphi(t)$. Thus for $a_0 = b_0 = c_0 d_0 = 0$ the solution is

$$\varphi(t) = \Pi(t - t_i)^{a_i}, \quad t_i = -\alpha/2 \pm (s^2(c - 2b) \pm (s^4 D - 4s^2\alpha b)^{1/2})^{1/2}$$

where a_i are coefficients obtained during integration.

In particular, for $\alpha = 0$ all the $a_i = -1/2$ and $\varphi(t) = [(t^2 - s^2 m_1)(t^2 - s^2 m_2)]^{-1/2}$.

Another special case of this equation, when all the coefficients c_{ik} are exponential functions of z, has already been examined /4/.

We also note that (1.9), meaning also its solution, is invariant under the substitution $c_1(z) = c(z) + 2(c_0 + c_1 z)$, $b_1(z) = b(z) + (c_0 + c_1 z)$. This enables us, using the above-mentioned substitution, to obtain new solutions by starting from known solutions.

In the case of degeneration of the roots of the characteristic equation $m_1^2 = m_2^2 = m_0^2$, new inhomogeneity laws can be added to those considered for which the solution can be obtained explicitly.

For $c' = 0$ the solution can be found in succession, first F_0 from (2.1) and then L_0 from (2.2).

For $m_0 = \mathrm{const}$ the substitution $x_1 = m_0 x_1$, $y = m_0 y_1$ reduces system (2.1) and (2.2) to an equation of an isotropic medium with variables z, x_1, y_1, where the parameters of the medium are $\mu = c^{-1}$, $c(1 - \nu) = m_0^2 d$. Consequently, all the solutions found earlier for an isotropic medium /5, 9/ will be valid even in this case.

We will show that the method /9, 10/ used earlier enables us to find exact solutions with variable m_0. We will use system (1.2) and (1.3). We will seek the solution in the form

$$F = F(z, s)s^{-1}\psi(x, y, z), \quad w = w(z, s)\psi(x, y, s)$$

where $\psi(x, y, s)$ satisfies (3.1).

We obtain a system of ordinary differential equations for $F(z, s)$ and $w(z, s)$ which we write in matrix form (E is the unit matrix)

$$\left[E\frac{d^2}{dz^2} - G\frac{d}{dz} - H\right]\left\|\begin{matrix}F\\ w\end{matrix}\right\| = 0 \tag{3.8}$$

$$G = -\left\|\begin{matrix}\dfrac{c_{44}'}{c_{44}} & \dfrac{c_{13} + c_{44}}{c_{44}}s \\ -\dfrac{c_{13} + c_{44}}{c_{33}}s & \dfrac{c_{33}'}{c_{33}}\end{matrix}\right\|, \quad H = \left\|\begin{matrix}\dfrac{c_{11}s^2}{c_{44}} & -\dfrac{c_{44}'}{c_{44}}s \\ \dfrac{c_{13}'}{c_{33}}s & \dfrac{c_{44}}{c_{33}}s^2\end{matrix}\right\|$$

The matrix Eq.(3.8) is factorized

$$\left[E\frac{d}{dz} - A\right]\left[E\frac{d}{dz} - B\right]\left\|\begin{matrix}F\\ w\end{matrix}\right\| = 0, \quad A = G - B, \quad \left[E\frac{d}{dz} - A\right]\psi_a = 0 \tag{3.9}$$

if the matrix B satisfies the Riccati equation $B' - GB + B^2 = H$. In this case the fundamental system of solutions of (3.8) ψ_b, $\psi_b\int\psi_b^{-1}\psi_a dz$ is expressed in terms of the fundamental solutions ψ_b, ψ_a of the corresponding equations. Taking into account that $G = G_0 + G_1 s$ and $H = H_1 s + H_2 s^2$ we will seek the solution in the form $B = B_1 s + B_0$ (the expansion in reciprocal powers of s is truncated). As in the case of the isotropic medium /9/, we obtain the system

$$G_1 B_1 + B_1^2 = H_2, \quad B_0 B_1 - (G_1 - B_1)B_0 = H_1 + G_0 B_1$$
$$B_0' - G_0 B_0 + B_0^2 = 0$$

If the solution B_1 of the first equation is selected such that the roots B_1 and $G_1 - B_1$ are identical

$$B_1 = \left\|\begin{matrix}0 & 1\\ m_0^2 & 0\end{matrix}\right\|, \quad G_1 - B_1 = \left\|\begin{matrix}0 & -m_0^2\dfrac{c_{33}}{c_{44}} \\ -\dfrac{c_{44}}{c_{33}} & 0\end{matrix}\right\|, \quad m_0^2 = \sqrt{\frac{c_{11}}{c_{33}}}$$

then the second equation reduces to the system (b_{ik} are the elements of the matrix B_0)

$$c_{44}b_{12} + c_{33}b_{21} = 0, \quad c_{44}b_{11} + m_0^2 c_{33} b_{22} = -2c_{44}' \tag{3.10}$$

The third equation has the solution $B_0^{-1} = \{K + \int M^{-1}dz\}M$ expressed in terms of a matrix of arbitrary constants K and the diagonal matrix $M = \mathrm{diag}\,\{c_{44}, c_{33}\}$.

It follows from the first equation in (3.10) that K is an antisymmetric matrix. The second equation governs the connection between c_{11}, c_{33} and c_{44} for whcih the mentioned solution holds ($\det K = A$, K_{ij} are elements of the matrix K)

$$m_0^2 = [k_{11}\alpha + 2c_{44}' (\Delta + k_{11}\alpha + k_{22}\beta + \alpha\beta)]/(k_{11} + \beta)$$
$$\alpha = \int c_{33}^{-1}dz, \ \beta = \int c_{44}^{-1}dz$$

Therefore, it can be concluded that 1) for $c_{44} = \text{const}$ we have the solution $B_0 = 0$ (all the coefficients except c_{44} are arbitrary functions), 2) the case when $b_{11} = -2c_{44}'/c_{44}$, the rest of the c_{ik} are arbitrary functions and 3) the case when $b_{22} = c_{33}^{-1} [k_{22} + \alpha]^{-1}$ the remaining $b_{ik} = 0$, corresponds to the dependence $c_{33} = x^2/x$, here $x = 2c_{44}'/m_0^2$.

Other simple solutions B_0 can be also be extracted. The factorization obtained enables us to find explicit solutions of (3.8) as in the case of an isotropic medium.

We will examine this in an example of the dependence $b_{22} = -x'/x = c_{33}^{-1} [k_{22} + \alpha]^{-1}$ (the third case) /10/.

The solution $\psi_b (F, w)$ is found from the equation $(Ed/dz - B)\psi_b = 0$ that reduces to the system

$$F' = sw; \ w' = m_0^2sw - x'w/x$$

Eliminating w we obtain $F'' + x'F'/x - m^2s^2F = 0$. The substitution $F = \varphi/\sqrt{x}$ reduces the equation to the normal form

$$\varphi'' - [m_0^2s^2 - D (1/x, z)/2] \varphi = 0 \tag{3.11}$$

The solution $\varphi = \eta (\zeta)(\zeta_z')^{-1/2}$ is related to the solution $\eta (\zeta)$ of the equation $\eta'' + P (\zeta) = 0$ (here $\zeta (z)$, $P (\zeta) = P_1 (\zeta) + P_2 (\zeta)$) for

$$-s^2m_0^2 (z)(z_\zeta')^2 = P_1 (\zeta) \tag{3.12}$$

$$\left[\frac{d^2}{d\zeta^2} + P_2 (\zeta)\right] \left[\frac{(c_{44})_z'}{m_0^2 (z) (z_\zeta')}\right]^{1/2} = 0 \tag{3.13}$$

By specifying the function $P (\zeta)$ corresponding to the known equations (Bessel or Whittaker) and representing it in the form of the sum $P_1 (\zeta), P_2 (\zeta)$ we find $m_0 (z)$ and c_{44} for which the solutions of (3.11) are expressed in terms of the solution of the equation selected in advance /10/.

Substituting $P_1 (\zeta)$ into (3.12) and solving this equation we find $\zeta (z)$ as a function of $m_0 (z)$ and $P_1 (\zeta)$ and we determine the appropriate value of $c_{44} (z)$ from (3.13).

Thus, the solution is expressed in terms of the exponential function for $P_1 (\zeta) = -s^2$

$$\zeta = \int m_0 (z) \, dz + c, \ \ P_2 (\zeta) = -n^2, \ \ c_{44} = (A_1e^{n\zeta} + A_2e^{-n\zeta})^2 + A_3$$

To find the fundamental system of equations it is also necessary to find $\psi_a (\varphi_1, \varphi_2)$ from system (3.9). It can be shown that system (3.9) reduces to (3.11) for $\varphi = \varphi_2c_{33}/\sqrt{x}$ i.e., ψ_a and ψ_b are actually found from one equation of (3.11).

As an illustration we will consider the Boussinesq problem for c_{44} and c_{13} constant, $c_{11} (z) = c_{11}z^2$ and $c_{33} (z) = c_{33}z^2$. We will seek the solution in the space of Hankel transforms

$$L (r, z) = \int_0^\infty L (z, s) J_0 (r, s) \, ds; \ \ F_0 (r, z) = \int_0^\infty F_0 (z, s) J_0 (r, s) \, ds.$$

The boundary conditions $\sigma_{xz} = \sigma_{yz} = 0$; $\sigma_{zz} = -P\delta (r) (2\pi r)^{-1}$ on the half-space surface are expressed for $z = z_0$ by means of L (1.8) and equal $L (z_0, s) = P/(2\pi s)$, $L' (z, s) = 0$ respectively, in the space of the Hankel transform. Here P is the concentrated force and $\delta (r)$ is the delta-function.

For the inhomogeneity under consideration, (3.2) reduces to Euler's equation $(n = -2)$. Taking account of the conditions at infinity $L (z, s) = L_1 (s) z^{\lambda_1} + L_2 (s) z^{\lambda_2}$ the solution of this equation is determined by the roots of (3.7).

Substituting $L (z, s)$ into the boundary conditions, we determine $L_1 (s), L_2 (s)$ and respectively

$$L (z, s) = \frac{P}{2\pi s (\lambda_2 - \lambda_1)} \left[\lambda_2 \left(\frac{z}{z_0}\right)^{\lambda_1} - \lambda_1 \left(\frac{z}{z_0}\right)^{\lambda_2}\right]$$

Since c, D are constants for the inhomogeneity law under consideration, we find from (1.11)

$$F_0 (z, s) = \frac{P\sqrt{D}}{4\pi s (\lambda_2 - \lambda_1)} \left[\lambda_2 \left(\frac{z}{z_0}\right)^{\lambda_1} + \lambda_2 \left(\frac{z}{z_0}\right)^{\lambda_2}\right]$$

The problem is completely defined by the two functions $L(z, s)$ and $F_0(z, s)$. Thus, we find from (1.4)

$$w(z, r) = -\left[\frac{1}{2c_{44}}\frac{\partial L}{\partial z} + \frac{\partial F_0}{\partial z}\right] = \int_0^\infty w(z, s) J_0(r, s)\, ds$$

$$w(z, s) = \frac{P}{4\pi} \frac{\lambda_1\lambda_2}{s(\lambda_1 - \lambda_2) z_0}\left[\left(\frac{z}{z_0}\right)^{\lambda_1 - 1}(\sqrt{D} + c) + \left(\frac{z}{z_0}\right)^{\lambda_2 - 1}(\sqrt{D} - c)\right]$$

In evaluating the integral it must be taken into account that $w(z_0, s)$ tends to a constant as $s \to \infty$, i.e., as $r \to 0$

$$w(z_0, r) = Pm_1m_2\sqrt{D}\,[2\pi(m_1 - m_2) r]^{-1} + \ldots$$

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